

STOCHASTIC FIELDS WITH PATHS IN ARBITRARY REARRANGEMENT INVARIANT SPACES.

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ABSTRACT.

We obtain sufficient conditions for belonging of almost all paths of a random process to some fixed rearrangement invariant (r.i.) Banach functional space, and to satisfying the Central Limit Theorem (CLT) in this space.

We describe also some possible applications.

Key words and phrases: Random process (field) (r.pr., r.f), path, rearrangement invariant (r.i.) Banach functional space, ball function, natural function and distance, Orlicz and Grand Lebesgue Spaces, separability, associate space, extremal points, Central Limit Theorem (CLT) in Banach space, Young-Fenchel, or Legendre transform, functional, metric entropy, lacunar trigonometrical series, majorizing measures, fundamental function, net, Rosenthal's inequality, Monte Carlo method, confidence region (c.r.).

1 Introduction. Notations. Statement of problem.

Let $(T = \{t\}, M, \mu)$ be measurable space with sigma - finite *separable* measure μ . Separability of the measure μ implies separability relative a distance

$$\rho(A_1, A_2) = \mu(A_1 \Delta A_2) = \mu(A_1 \setminus A_2) + \mu(A_2 \setminus A_1).$$

Let $(L, \|\cdot\|_L)$ be some fixed *separable* Banach functional rearrangement invariant (r.i.) space over the triple (T, M, μ) . We refer the readers about these definitions to the famous book [1], chapters 1,2.

Let also $\xi(t) = \xi(t, \omega)$, $t \in T$ be *separable* numerical valued (real or complex) random process (r.pr.) or random field, defined aside from the source triplet on some probability space (Ω, B, \mathbf{P}) with expectation \mathbf{E} and variance Var .

We raise the question: under what the sufficient conditions almost all the paths of the r.pr. $\xi(t)$ belong to the space L :

$$\mathbf{P}(\xi(\cdot) \in L) = 1? \tag{1.1}$$

A second question: under what the sufficient conditions the r.f. $\xi(t)$ not only belongs to the space L a.e., but in addition satisfies the CLT in this space?

Recall that by definition the r.f. $\xi(t)$ satisfies CLT in some Banach functional space B , iff the suitable normed sums

$$S_n(t) := n^{-1/2} \sum_{i=1}^n \xi_i(t),$$

where $\{\xi_i(t)\}$ are independent copies of r.f. $\xi(t)$, converges weakly (in distribution) in the space B as $n \rightarrow \infty$ to a non-degenerate Gaussian random field $S_\infty(t) : \text{Law}(S_n) \rightarrow \text{Law}(S_\infty)$. In detail: for arbitrary continuous bounded functional $F : L \rightarrow R$

$$\lim_{n \rightarrow \infty} \mathbf{E}F(S_n(\cdot)) = \mathbf{E}F(S_\infty). \quad (1.2)$$

In particular, if $\text{Law}(S_n) \xrightarrow{\text{def}} \text{Law}(S_\infty)$, then $\forall u = \text{const} > 0 \Rightarrow$

$$\lim_{n \rightarrow \infty} \mathbf{P}(\|S_n\|B > u) = \mathbf{P}(\|S_\infty\|B > u). \quad (1.3)$$

Evidently, if $\xi(t)$ satisfies the CLT in Banach functional space, then $\mathbf{E}\xi(t) = 0$, $\mathbf{E}\xi^2(t) < \infty$, $t \in T$, and the r.f. $\xi(t)$ is pregaussian. This means by definition that the Gaussian centered r.f. $S_\infty(t)$, which has at the same covariation function as $\xi(t)$:

$$R(t, s) := \mathbf{E}S_\infty(t)S_\infty(s) = \mathbf{E}S_n(t)S_n(s) = \mathbf{E}\xi(t)\xi(s) \quad (1.4)$$

belongs to the space B with probability one.

Many sufficient conditions for the equality $\mathbf{P}(\xi(\cdot) \in B) = 1$ for different separable Banach spaces B are obtained in [26], [19], [31], [5], [6], [29], [30], [32], [9] - [12]. The case of rearrangement invariant spaces, especially ones exponential type Orlicz's spaces, is considered in the articles [13], [14], [15].

The sufficient conditions for CLT in the Banach space of continuous functions may be found in [16], [18], [19], [20], [22], [23], [24] etc. CLT in another separable Banach spaces is investigated, e.g. in [17], [21], [23], [19], [31], [29], [30], [32], [33]. The article [14] is devoted to the CLT in the exponential Orlicz space, more exactly, to the CLT in some separable subspace of the exponential type Orlicz space.

The technology of application of the Banach space valued Central Limit Theorem in the parametric Monte Carlo method is described in [34], [35], [3], [36].

WE NEED TO INTRODUCE SOME NEW NOTATIONS.

A. Associate space. We denote by L' the *associate* space to the source space L , i.e. the set of all continuous (bounded) linear functionals of the form

$$l_g(f) = \int_T f(t) g(t) \mu(dt)$$

with finite ordinary norm

$$\|g\|_{L'} = \sup_{f: f \in L, f \neq 0} \left[\frac{l_g(f)}{\|f\|} \right] = \sup_{f: f \in L, \|f\|=1} l_g(f). \quad (1.5)$$

It is known that $(L')' = L$, see [1], chapter 2.

Denote by $S = S_e$ the set of all *extremal* points of the unit surface of associate space L' , so that $S = S_e$ is symmetric: $-S = S$ and $\forall g \in S \Rightarrow \|g\|_{L'} = 1$; then

$$\forall f \in L \Rightarrow \|f\|_L = \sup_{g \in S_e} l_g(f). \quad (1.6)$$

B. Fundamental function. Recall that the fundamental function $\phi(L, \delta)$, $\delta = \text{const} > 0$ for r.i. Banach functional space $(L, \|\cdot\|_L)$ is defined as follows. Let A be some measurable set with measure $\delta : \mu(A) = \delta > 0$. Then

$$\phi(L, \delta) \stackrel{\text{def}}{=} \|I(A)\|_L; \quad (1.7)$$

in the sequel $I(A) = I_A(t)$ denotes an indicator function of the set A . This definition does not dependent on the concrete representation of the set A and play a very important role in the theory of operators, theory of Fourier series etc. see [1], chapters 4,5.

Note that $\phi(L, \delta) \cdot \phi(L', \delta) = \delta$.

C. Metric entropy.

Further, let (X, ρ) be compact metric space relative the distance (or semi - distance) function $r = r(x_1, x_2)$. Denote as usually by $N(X, r, \epsilon)$, $\epsilon = \text{const} > 0$ the minimal number of closed r - balls of a radii ϵ which cover all the set X ; obviously, $\forall \epsilon > 0 \Rightarrow N(X, r, \epsilon) < \infty$.

The quality

$$H(X, r, \epsilon) = \ln N(X, r, \epsilon) \quad (1.8)$$

is named *entropy* of the set X relative the distance r at the point ϵ .

This notion is in detail investigated, e.g., in [3], chapter 3, section 3.2.

D. Grand Lebesgue spaces.

Recently, see [38], [39], [40], [41], [42], [26], [3] etc. appears the so-called *Grand Lebesgue Spaces* $GLS = G(\psi) = G(\psi; a, b)$, $a, b = \text{const}, a \geq 1, a < b \leq \infty$, spaces consisting on all the measurable functions $f : T \rightarrow R$, where (recall) $(T = \{t\}, M, \mu)$ is measurable space with non - trivial sigma - finite measure μ , having a finite norms

$$\|f\|_{G(\psi)} = \|f\|_{G(\psi; \mu)} \stackrel{\text{def}}{=} \sup_{p \in (a, b)} [\|f\|_{p, \mu} / \psi(p)], \quad (1.9)$$

where we define as usually

$$\|f\|_{p, \mu} := \left[\int_T |f(t)|^p \mu(dt) \right]^{1/p}, \quad 1 \leq p < \infty.$$

and we define correspondingly for a random variable η

$$|\eta|_p := [\mathbf{E}|\eta|^p]^{1/p}.$$

Here $\psi(\cdot)$ is some continuous positive on the *open* interval (a, b) function such that

$$\inf_{p \in (a, b)} \psi(p) > 0.$$

We can suppose without loss of generality

$$\inf_{p \in (a, b)} \psi(p) = 1.$$

Notation: $(a, b) = \text{supp } \psi$.

As the capacity of the measure μ may be picked the probability measure \mathbf{P} defined on all the measurable sets B ; $B \subset \Omega$.

This spaces are rearrangement invariant, see [1], and are used, for example, in the theory of probability [9], [26], [3]; theory of Partial Differential Equations [39], [42]; functional analysis; theory of Fourier series; theory of martingales etc.

Let $\delta = \text{const} > 0$; the fundamental function $\phi(\delta) = \phi_{G(\psi)}(\delta)$ of the space $G(\psi)$ may be calculated as follows:

$$\phi_{G(\psi)}(\delta) = \sup_{p \in (a, b)} \left[\frac{\delta^{1/p}}{\psi(p)} \right].$$

The fundamental function for GLS is in detail investigated in [43]; in particular, it was therein calculated many examples. Roughly speaking, there is an essential difference for behavior of this function between the cases $b < \infty$ and $b = \infty$.

D. Definition 1.1. Suppose the set T is equipped in addition with some distance (semi-distance) function $d = d(t, s)$ such that sigma field M is Borelian sigma algebra (metric measurable space) and such that the metric space (T, d) is compact space.

The *ball function* $r(\delta) = r(T, \delta)$, $\delta = \text{const} > 0$ for the measurable space (T, M, μ) equipped with a distance $d = d(t, s)$ is by definition the function of a form

$$r(T, d, \delta) = r(T, \delta) = r(\delta) \stackrel{\text{def}}{=} \sup_{t \in T} \mu[B(t, \delta)], \quad (1.10)$$

where as ordinary

$$B(t, \delta) = B(t, d, \delta) = \{z, z \in T, d(t, z) \leq \delta\}$$

is closed δ – ball in the metric space (T, d) with the center at the point t , $t \in T$.

It is clear that $\lim_{\delta \rightarrow 0+} r(T, \delta) = 0$.

Suppose now the measure μ in is probabilistic: $\mu(T) = 1$, and let the function $\psi = \psi(p)$ be such that $\text{supp } \psi = (1, \infty)$. Then the Grand Lebesgue Space $G(\psi)$ coincides up to norm equivalence to the subspace of all mean zero: $\int_T f(t) \mu(dt) = 0$

measurable function (random variables) of the so-called *exponential Orlicz space* $L(N) = L(N; T, \nu)$ with *exponential* Orlicz-Young function $N = N(u)$, and conversely proposition is also true: arbitrary exponential Orlicz space $L(N)$ coincides with some Grand Lebesgue Space, see [26].

In detail, introduce the function $\phi(\cdot)$ as follows:

$$\chi(p) := p \psi(p), \quad p \geq 2, \quad \chi(p) = C \cdot p^2, \quad 0 \leq p \leq 2, \quad C : 2\chi(2) = 4C,$$

$$\phi(y) := (\chi(y))^{-1}, \quad y \geq 0; \quad \phi(y) := \phi(|y|), \quad y < 0;$$

then

$$N(u) = \exp(\phi^*(u)) - 1,$$

where $\phi^*(\cdot)$ is Young - Fenchel, or Legendre transform for the function ϕ :

$$\phi^*(u) = \sup_{y \in R} (uy - \phi(y)),$$

see [3], chapter 1, theorem 1.5.1.

The finiteness of the $G\psi$ norm for the r.v. ξ allows to obtain the quasy - exponential bounds for its tail of distribution. Indeed, if we denote

$$T_\xi(x) = \max [\mathbf{P}(\xi > x), \mathbf{P}(\xi < -x)], \quad x \geq 2,$$

and if $0 < \|\xi\|G\psi < \infty$, then

$$T_\xi(x) \leq \exp \{ -[p \ln \psi(p)]^* (\ln x / \|\xi\|G\psi) \}, \quad x \geq 2\|\xi\|G\psi,$$

and the conversely proposition is true. Namely, if for some r.v. ξ

$$T_\xi(x) \leq \exp \{ -h(\ln x) \}, \quad x \geq 2,$$

where $h = h(y)$ is positive continuous convex strictly monotonically increasing function such that $\lim_{y \rightarrow \infty} h(y) = \infty$, then

$$\|\xi\|G\psi \leq C(h) < \infty, \quad \textbf{where} \quad \psi(p) := \exp \left(\frac{h^*(p)}{p} \right).$$

Let for instance $m = \text{const} > 0$ and define the following ψ function

$$\psi_m(p) = p^{1/m}, \quad 1 \leq p < \infty.$$

The r.v. ψ belongs to the space $G\psi_m$ iff for some positive constant $C(m)$

$$T_\xi(x) \leq \exp (-C(m) x^m), \quad x \geq 2.$$

The last proposition is well known, see e.g. [14].

The case $m = 2$ correspondent to the case of the so-called subgaussian random variables, centered or not. Here $\psi(p) = \sqrt{p}$.

Let now $F = \{f_\alpha(t)\}$, $\alpha \in A$ be a *family* of measurable functions such that

$$\exists(a, b), 1 \leq a < b \leq \infty, \quad \forall p \in (a, b) \Rightarrow \sup_{\alpha \in (a, b)} \|f_\alpha\|_{p, \mu} < \infty. \quad (1.11)$$

The function

$$\psi_F(p) := \sup_{\alpha \in (a, b)} \|f_\alpha(\cdot)\|_{p, \mu} \quad (1.12)$$

is said to be *natural function* for the family F . This function is obviously minimal up to equivalence function ψ for which

$$\sup_{\alpha \in A} \|f_\alpha\|_{G\psi} = 1.$$

2 First condition.

We do not assume in this and in the next sections that the r.i. space L is separable.

Suppose here that the r.f. $\xi(t)$ belongs uniformly in t , $t \in T$ to some non-trivial $G\psi_0$ space:

$$\exists a, b : 1 = a < b \leq \infty, \Rightarrow \forall p \in (a, b) \quad \psi_0(p) := \sup_{t \in T} \|\xi(t)\|_{L(p), \Omega} < \infty. \quad (2.1)$$

In what follow we can use instead the natural function ψ_0 in (2.1) arbitrary its majorant $\psi = \psi(p)$ from the set $G\Psi$ with at the same support (a, b) .

Let us introduce a so-called *natural*, i.e. generated only by means of the values of the r.f. $\{\xi(t)\}$, $t \in T$, on the set T *bounded* semi-distance $d_\psi = d_\psi(t, s)$ as follows

$$d_\psi(t, s) = d_\psi \stackrel{\text{def}}{=} \|\xi(t) - \xi(s)\|_{G\psi}. \quad (2.2)$$

Note that for natural distance d_{ψ_0} $d_{\psi_0}(t, s) \leq 2$ and that the r.f. $\xi(t)$ is stochastic continuous relative this distance.

Theorem 2.1. *Suppose that for some $q = \text{const} \in (0, 1)$ the following entropy series converge:*

$$\sigma = \sigma(q) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} q^n N(T, d_\psi, q^{n+1}) \quad r(T, q^n) < \infty. \quad (2.3)$$

Then

$$\mathbf{P}(\xi(\cdot) \in L) = 1 \quad (2.4)$$

and moreover if in addition $\sup_{t \in T} \|\xi(t)\|_{G\psi} = 1$, then

$$\|\|\xi(\cdot)\|_L\|_{G\psi} \leq \underline{\sigma} := \inf_{q \in (0, 1)} \left\{ \sum_{n=0}^{\infty} q^n N(T, d_\psi, q^{n+1}) \quad r(T, q^n) \right\}. \quad (2.5)$$

Proof. We can and will assume without loss of generality $\sigma = 1$. Further, denote by $T_n = T_n(q^n)$ the minimal q^n of the set T relative the distance d_ψ .

This net is not necessary to be unique; we pick arbitrary fixed but non-random one.

We have on the basis of entropy definition

$$\text{card}(T_n) = N(q^{n+1}) := N(T, d_\psi, q^{n+1}). \quad (2.6)$$

We define for arbitrary element $t \in T$ and any value $n = 0, 1, 2, \dots$ the following "projection" $\theta_n(t)$:

$$d_\psi(t, \theta_n(t)) \leq q^n, \quad \theta_n(t) \in T_n. \quad (2.7)$$

This point may be also not unique, but we choose it non - random. By definition, $\theta_0(t) := t_0 \in T_0$ be some fixed point inside the set T .

We have

$$\xi(t) = \sum_{n=0}^{\infty} (\xi(\theta_{n+1}(t)) - \xi(\theta_n(t))), \quad (2.8)$$

therefore

$$\|\xi(\cdot)\|L \leq \sum_{n=0}^{\infty} \eta_n, \quad \eta_n := \|\xi(\theta_{n+1}(\cdot)) - \xi(\theta_n(\cdot))\|L. \quad (2.9)$$

The function $t \rightarrow \xi(\theta_{n+1}(\cdot)) - \xi(\theta_n(\cdot))$ is simple (stepwise), therefore it belongs to the space L . The amount of the d_ψ balls of a radii q^{n+1} is less or equal than $N(q^{n+1})$. The value $|\xi(\theta_{n+1}(\cdot)) - \xi(\theta_n(\cdot))|$ does not exceed the value q^n .

Since the L - space is rearrangement invariant,

$$\|\eta_n\|G\psi \leq q^n N(q^{n+1}) r(T, q^n). \quad (2.10)$$

It remains to use the triangle inequality and completeness of the r.i. Banach functional space L :

$$\|\|\xi(\cdot)\|L\|G\psi \leq \sum_{n=0}^{\infty} q^n N(q^{n+1}) r(T, q^n) = \sigma(p).$$

Since the value q is arbitrary inside the interval $(0, 1)$,

$$\mathbf{P}(\xi(\cdot) \in L) = 1, \quad \|\|\xi(\cdot)\|L\|G\psi \leq \inf_{q \in (0,1)} \sigma(q) = \underline{\sigma},$$

Q.E.D.

As a consequence: under formulated above conditions $\mathbf{P}(\xi(\cdot) \in L) = 1$ and moreover

$$\mathbf{P}(\|\xi(\cdot)\|L > x) \leq \exp \{-[p \ln \psi(p)]^* (\ln x / \underline{\sigma})\}, \quad x \geq 2\underline{\sigma}. \quad (2.11)$$

Remark 2.1. The expression $\|\|\xi(\cdot)\|L\|G\psi$ is called mixed, or on the other words Bochner's norm for the function of two variables (random process) $\xi = \xi(t, \omega)$.

Examples.

First example.

Conditions. Let $\psi = \psi(p)$ be some non-trivial: $b = \sup\{p; p \in \text{supp } \psi\} > 1$ natural function for the r.f. $\xi(t) : \sup_t \|\xi(t)\|_G \psi = 1$. Then evidently $d(t, s) = \|\xi(t) - \xi(s)\|$.

Suppose first of all that $r(T, \delta) \leq \delta^s, \delta \in (0, 1)$ for some positive value $s = \text{const} > 0$.

If for instance T is closure of an open set in the space R^d and $r(t, s) \asymp |t - s|^\alpha$, where $|t|$ is usually Euclidean norm and $\alpha = \text{const} \in (0, 1]$, then $s = d/\alpha$.

Assume further

$$N(T, d_\psi, \epsilon) \leq \epsilon^{-\kappa}, \epsilon \in (0, 1), \kappa = \text{const} \in (0, 1 + s). \quad (2.12)$$

The last equality (2.12) implies that the entropy dimension of the set d relative the distance d_ψ is restricted

$$\dim_{d_\psi}(T) = \kappa < 1 + s.$$

In the sequel we assume the values κ, s to be fixed. Another notations: $\Delta = 1 + s - \kappa > 0, \lambda = \kappa/\Delta$.

We deduce that all the conditions of theorem 2.1 are satisfied and we obtain after some calculations that the optimal value of the parameter p is following:

$$p_0 = \left(\frac{\lambda}{1 + \lambda} \right)^{1/\Delta} = \left[\frac{\kappa}{\kappa + \Delta} \right]^{1/\Delta}$$

and correspondingly

$$\underline{\sigma} = \left[\frac{\kappa}{\Delta} \right]^{-\kappa/\Delta} \cdot \left[1 + \frac{\kappa}{\Delta} \right]^{-1-\kappa/\Delta}. \quad (2.13)$$

Second example.

Al the parameters are as before in the first example aside from the entropy condition:

$$N(\epsilon) \leq \epsilon^{-(1+s)} |\ln \epsilon|^{-\beta}, \epsilon \in (0, 1/e), \beta = \text{const} > 1, \kappa = 1 + s. \quad (2.14)$$

We deduce again that all the conditions of theorem 2.1 are satisfied and we obtain after some calculations that the optimal value of the parameter p is following:

$$p_0 = \exp(-\beta/\kappa)$$

and correspondingly

$$\underline{\sigma} \leq e^{-\beta} \beta^\beta \kappa^{-\beta} (\zeta_R(\beta) - 1), \quad (2.15)$$

where $\zeta_R(\cdot)$ denotes ordinary Rieman's zeta function.

3 Second condition.

Since the r.f. $\xi(t)$ is stochastic continuous, it is (T, M) measurable.

We suppose in addition that for arbitrary (non-random!) function $g = g(t)$ from the space L' there exists (with probability one) the following linear functional (integral):

$$\forall g \in L' \quad \exists l_\xi(g) := \int_T \xi(t) g(t) \mu(dt). \quad (3.1)$$

A simple sufficient condition for (3.1) is following: the function $t \rightarrow \mathbf{E}|\xi(t)|$ there exists and belongs to the space L .

It remains to establish the finiteness with probability one the value

$$\nu := \sup_{g: \|g\|_{L'}=1} l_\xi(g).$$

As long as the set $S = S_e$ is the set of all extremal points in the centered unit ball of the space L' ,

$$\nu = \sup_{g \in S} l_\xi(g). \quad (3.2)$$

In what follow we consider in this section only the case when $g(\cdot) \in S$.

Suppose that as in the last section that the family of random variables $\{\xi(t)\}$, $t \in T$ obeys some non - trivial natural ψ function:

$$\sup_{t \in T} \|\xi(t)\| G\psi = 1, \quad b := \sup \sup \psi > 1. \quad (3.3)$$

We estimate using triangle (Marcinkiewicz) inequality

$$\begin{aligned} \|l_\xi(g)\| G\psi &= \left\| \int_T \xi(t) g(t) \mu(dt) \right\| G\psi \leq \int_T \|\xi(t)\| G\psi |g(t)| \mu(dt) \leq \\ &\int_T |g(t)| \mu(dt) = \|g(\cdot)\|_{L_1(T, \mu)} \leq C_1 = \text{const} < \infty. \end{aligned} \quad (3.4)$$

We introduce now the (semi-) distance $\rho(g_1, g_2)$ on the set S_e as follows:

$$\rho(g_1, g_2) := \|g_1(\cdot) - g_2(\cdot)\|_{L_1(T, \mu)}.$$

We have analogously

$$\begin{aligned} \|l_\xi(g_1) - l_\xi(g_2)\| G\psi &= \left\| \int_T \xi(t) (g_1(t) - g_2(t)) \mu(dt) \right\| G\psi \leq \\ &\int_T \|\xi(t)\| G\psi |g_1(t) - g_2(t)| \mu(dt) = \|g_1 - g_2\|_{L_1(T, \mu)} = \rho(g_1, g_2). \end{aligned} \quad (3.5)$$

Define also for arbitrary function $f : R_+ \rightarrow R$ the Young-Fenchel co-transform f_* by an equality

$$f_*(x) \stackrel{\text{def}}{=} \inf_{y \geq 0} (xy + f(y)), \quad (3.6)$$

and introduce the diameter of the set S relative the semi-distance $\rho(\cdot, \cdot)$ $D := \sup_{g_1, g_2 \in S} \rho(g_1, g_2) < \infty$ and a function $v(y) = \ln \psi(1/y)$.

Theorem 3.1. *Suppose in addition to the formulate above conditions*

$$I := \int_0^D \exp(v_*(2 + \ln N(T, d_\psi, \epsilon))) d\epsilon < \infty. \quad (3.7)$$

Then $\mathbf{P}(\xi(\cdot) \in L) = 1$.

Proof follows immediately from the theorem 3.17.1 of the monograph [3], chapter 3, section 3.17, where is proved in particular that

$$(\|\xi\|_L \|G\psi =) \|\nu\|G\psi = \|\sup_{g \in S} l_\xi(g)\|G\psi \leq 9I < \infty. \quad (3.8)$$

This completes the proof of theorem 3.1.

4 Conditions for the Central Limit Theorem.

We suppose in this section that the r.i. space L is separable.

Suppose in addition that the random field $\xi(t)$ is mean zero, has uniform bounded second moment and is pregaussian.

First version.

Suppose here in this subsection as before that the r.f. $\xi(t)$ belongs uniformly in t , $t \in T$ to some non - trivial $G\psi_0$ space:

$$\exists a, b : 2 = a < b \leq \infty, \Rightarrow \forall p \in (a, b) \psi_0(p) := \sup_{t \in T} \|\xi(t)\|_{L(p), \Omega} < \infty. \quad (4.1)$$

In what follow we can use instead the natural function ψ_0 in (4.1) arbitrary its majorant $\psi = \psi(p)$ from the set $G\Psi$ with at the same support (a, b) .

We define for arbitrary such a function $\psi(\cdot)$ its *Rosenthal's* transform $\psi_R(\cdot)$:

$$\psi_R(p) \stackrel{\text{def}}{=} \frac{C_R p}{\ln p} \cdot \psi(p), \quad p \in (2, b), \quad C_R := 1.77638. \quad (4.2)$$

It is clear that if $b < \infty$, then $\psi_R(\cdot) \asymp \psi(\cdot)$, $1 \leq p < b$. Therefore, we will assume in this approach $b = \infty$.

The classical Rosenthal's inequality [46] asserts in particular that if $\{\zeta_i\}$, $i = 1, 2, \dots$ are the sequence of i., i.d. centered r.v. with finite p^{th} moment, then

$$\sup_n \left| n^{-1/2} \sum_{i=1}^n \zeta_i \right|_p \leq \frac{C_R p}{\ln p} |\zeta_1|_p, \quad p \geq 2. \quad (4.3)$$

About the exact value of the constant C_R see the article [44]. Note that for symmetrical distributed r.v. $C_R \leq 1.53573$.

Let us consider the normed sums

$$S_n(t) := n^{-1/2} \sum_{i=1}^n \xi_i(t), \quad n = 1, 2, \dots \quad (4.4)$$

It follows from Rosenthal's inequality

$$\sup_n \sup_{t \in T} \|S_n(t)\| G\psi_R \leq 1, \quad (4.5a)$$

and we define

$$\rho_\psi(t, s) \stackrel{\text{def}}{=} \sup_n \|S_n(t) - S_n(s)\| G\psi_R. \quad (4.5b)$$

Theorem 4.1. *Suppose that for some $q = \text{const} \in (0, 1)$ the following entropy series converge:*

$$\gamma = \gamma(q) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} q^n N\left(T, \rho_\psi, q^{n+1}\right) \quad r(T, \rho_\psi, q^n) < \infty. \quad (4.6)$$

Then the sequence of r.f. $\xi_i(t)$, $t \in T$ satisfies the CLT in the r.i. L space.

Proof. The convergence of finite dimensional (cylindrical) distributions of r.f. $S_n(t)$ to the finite dimensional of the Gaussian r.f. $S_\infty(t)$, which sample path in turn belongs to the space L is evident. It remains to establish the weak compactness of measures in the space L generated by r.f. $S_n(\cdot)$.

We apply theorem 2.1 to the random field $S_n(\cdot)$ for arbitrary fixed value q for which $\gamma(q) < \infty$:

$$\sup_n \| \|S_n(\cdot)\|_L \| G\psi_R \leq \gamma(q) < \infty. \quad (4.7)$$

Further, as long as the space L is presumed to be separable, there exists a *compact* linear operator $U : L \rightarrow L$ such that $U^{-1}\xi \in L, \Rightarrow U^{-1}S_n \in L$ and moreover

$$\sup_n \| \|U^{-1}S_n(\cdot)\|_L \| G\psi_R \leq 1, \quad (4.8)$$

see [37], [45]. Therefore,

$$\lim_{Z \rightarrow \infty} \sup_n \mathbf{P} \left(\|U^{-1}S_n\|_L > Z \right) = 0. \quad (4.9)$$

Since the set

$$W(Z) = \{f, f \in L, \quad \|U^{-1}f\|_L \leq Z\}$$

is compact subset of the space L , the equality (4.9) proves the proposition of theorem 4.1.

See also the criterion for the functional CLT in the famous book [19], chapter 6.

Second version.

We intent to mention and to generalise the third our section.

We suppose as before in addition that for arbitrary (non - random!) function $g = g(t)$ from the space L' there exists (with probability one) the following linear functional (integral):

$$\forall g \in L' \quad \exists l_\xi(g) := \int_T \xi(t) g(t) \mu(dt).$$

Then automatically

$$\forall g \in L' \quad \exists l_{S_n}(g) := \int_T S_n(t) g(t) \mu(dt).$$

It remains only to establish the finiteness with probability one the value

$$\lambda := \sup_n \sup_{g: \|g\|_{L'}=1} l_{S_n}(g).$$

or equally

$$\lambda = \sup_{g \in S} l_{S_n}(g).$$

In what follow we consider in this section only the case when $g(\cdot) \in S$.

Suppose that as in the third section that the family of random variables $\{\xi(t)\}$, $t \in T$ obeys some non-trivial natural ψ function:

$$\sup_{t \in T} \|\xi(t)\| G\psi = 1, \quad b := \sup \sup \psi > 1.$$

Then

$$\sup_{t \in T} \|S_n(t)\| G\psi_R = 1, \quad b := \sup \sup \psi = \infty,$$

the case $b < \infty$ is trivial.

We estimate using triangle (Marcinkiewicz) inequality

$$\|l_{S_n}(g)\| G\psi_R = \left\| \int_T S_n(t) g(t) \mu(dt) \right\| G\psi_R \leq \int_T \|S_n(t)\| G\psi_R |g(t)| \mu(dt) \leq$$

$$\int_T |g(t)| \mu(dt) = \|g(\cdot)\|_{L_1(T, \mu)} \leq C_1 = \text{const} < \infty. \quad (4.10)$$

Recall that we introduced the (semi-) distance $\rho(g_1, g_2)$ on the set $S = S_e$ as follows:

$$\rho(g_1, g_2) := \|g_1(\cdot) - g_2(\cdot)\|_{L_1(T, \mu)}.$$

We have analogously

$$\|l_{S_n}(g_1) - l_{S_n}(g_2)\|_{G\psi_R} = \left\| \int_T S_n(t) (g_1(t) - g_2(t)) \mu(dt) \right\|_{G\psi_R} \leq$$

$$\int_T \|S_n(t)\|_{G\psi_R} |g_1(t) - g_2(t)| \mu(dt) = \|g_1 - g_2\|_{L_1(T, \mu)} = \rho(g_1, g_2). \quad (4.11)$$

and introduce the diameter of the set S relative the semi-distance $\rho(\cdot, \cdot)$ $D := \sup_{g_1, g_2 \in S} \rho(g_1, g_2) < \infty$ and a function $v_R(y) = \ln \psi_R(1/y)$.

Theorem 4.2. *Suppose in addition to the formulate above conditions*

$$J := \int_0^D \exp(v_{R,*}(2 + \ln N(T, d_{\psi_R}, \epsilon))) d\epsilon < \infty. \quad (4.12)$$

Then the sequence of the r.f. $S_n(\cdot)$ satisfies the CLT in the space L .

Proof follows immediately from the theorem 3.17.1 of the monograph [3], chapter 3, section 3.17, where it is proved in particular that uniformly in n

$$\left(\sup_n \|S_n\|_L \|G\psi_R\| \right) \|\lambda\|_{G\psi_R} = \sup_n \left\| \sup_{g \in S} l_{S_n}(g) \right\|_{G\psi_R} \leq 9J < \infty.$$

This completes the proof of theorem 4.2.

5 Concluding remarks. Applications.

A. Applications in the Monte-Carlo method.

Let us consider here the problem of Monte-Carlo approximation and construction of a confidence region in the L – space norm for the parametric integral of a view

$$I(t) = \int_X g(t, x) \nu(dx). \quad (5.1)$$

Here (X, F, ν) is also a probabilistic space with normed: $\nu(X) = 1$ non-trivial measure ν .

A so-called Depending Trial Method estimation for the integral (5.1) was introduced by Frolov A.S. and Tchentsov N.N., see [34]:

$$I_n(t) = n^{-1} \sum_{i=1}^{\infty} g(t, \eta_i), \quad (5.2)$$

where $\{\eta_i\}$ is the sequence of ν distributed: $\mathbf{P}(\eta_i \in A) = \nu(A)$ independent random variables.

Suppose that the sequence of r.f. $g(t, \eta_i) - I(t)$ satisfies the CLT in some Banach r.i. space L ; then

$$\lim_{n \rightarrow \infty} \mathbf{P} \left(\sqrt{n} \|I_n(\cdot) - I(t)\|_L > u \right) = \mathbf{P}(\|\zeta(\cdot)\|_L > u), \quad u > 0; \quad (5.3)$$

therefore

$$\mathbf{P} \left(\sqrt{n} \|I_n(\cdot) - I(t)\|_L > u \right) \approx \mathbf{P}(\|\zeta(\cdot)\|_L > u), \quad u > 0; \quad (5.4)$$

The last equality may be used by the construction of a confidence region (c.r.) in the L norm for the integral $I(t)$. Namely, equating the right - hand side of (5.3a) to some "small" number δ , for instance $\delta = 0.05$ or $\delta = 0.01$ etc., where the value $1 - \delta$ is reliability of the c.r.:

$$\mathbf{P}(\|\zeta(\cdot)\|_L > u_0) = \delta, \quad (5.5)$$

we obtain an asymptotical c. r. of a form: with probability $\approx 1 - \delta$

$$\|I_n(\cdot) - I(t)\|_L \leq \frac{u_0}{\sqrt{n}}. \quad (5.6)$$

See for detail description the articles [35], [28], [7], [8].

B. A case of Hölder - Lipshitz space.

The CLT in the so-called Hölder (Lipshitz) space $H^o(\omega)$, (but which is not rearrangement invariant), is investigated, e.g. in [3], chapter 4, section 4.13.

Recall that the Hölder Lipshitz space $H^o(\omega)$ consists on all the numerical continuous relative some distance $d = d(t, s)$ functions $f : T \rightarrow R$ satisfying the condition

$$\lim_{\delta \rightarrow 0+} \frac{\omega(f, \delta)}{\omega(\delta)} = 0. \quad (5.7)$$

Here $\omega(f, \delta)$ is uniform module of continuity of the function f :

$$\omega(f, \delta) = \sup_{t, s: d(t, s) \leq \delta} |f(t) - f(s)|,$$

$\omega(\delta)$ is *some* non - trivial (continuous) module of continuity. For example, $\omega(0+) = \omega(0) = 0$, $\delta > 0 \Rightarrow \omega(\delta) > 0$ etc.

The metric space (T, d) is presumed to be compact.

The norm of the space $H^o(\omega)$ is defined as follows:

$$\|f\|_{H^o(\omega)} = \sup_{t \in T} |f(t)| + \sup_{\delta \in (0, 1)} \omega(f, \delta). \quad (5.8)$$

This modification of the classical Lipshitz space is in general case separable.

The recent version for CLT in Hölder spaces, for example for the Banach space valued random processes, see in [32].

In the article of B.Heinkel [24] is obtained sufficient condition for CLT in the space of continuous functions $C(T, d)$ in the more modern terms of "majorizing measures"; see [2], [9] - [12].

It is interest by our opinion to obtain the conditions for CLT in these terms for the Hölder-Lipshitz spaces, as well as for the separable functional rearrangement invariant spaces.

C. Counterexample.

Let $T = [0, 2\pi]$. There exists an example of mean zero *continuous* periodical r.pr. $\xi(t)$ constructed by means of lacunar trigonometrical series which does not satisfy the CLT in the space $C(T)$, see [26]. The analog of the conditions of theorem 3.1 for this space is satisfied but the conditions of theorem 4.2 are not.

This process can serve as an example (counterexample) to our situation. More detail, let us consider the Orlicz space \hat{L} over the set T with the Young-Orlicz function $N(u) = \exp(u^4) - 1$. As we know, the norm in this space may be defined up to equivalence as follows:

$$\|f\|_{\hat{L}} := \sup_{p \geq 1} \left[\frac{|f|_p}{p^{1/4}} \right].$$

But this space is not separable. In order to obtain the separable space, we introduce as a capacity of the space L the subspace of \hat{L} consisting on all the function $f \in \hat{L}$ for which

$$\lim_{p \rightarrow \infty} \left[\frac{|f|_p}{p^{1/4}} \right] = 0.$$

As long as the limiting Gaussian process $S_\infty(t)$, $t \in T$ described in third section does not belongs to the space L , the continuous a.e. r.pr. $\xi(t)$ does not satisfy the CLT also in the space L .

References

- [1] BENNET G., SHARPLEY R. *Interpolation of operators*. Orlando, Academic Press Inc., (1988).
- [2] FERNIQUE X. *Regularite de fonctions aleatoires non gaussiennes*. Ecole de Ete de Probabilits de Saint-Flour XI-1981. Lecture Notes in Math. 976, (1983), 174, Springer, Berlin
- [3] OSTROVSKY E.I. *Exponential Estimations for Random Fields*. Moscow-Obninsk, OINPE, (1999), in Russian.
- [4] OSTROVSKY E., SIROTA L. *Moment Banach spaces: Theory and applications*. HAIT Journal of Science and Engineering C, Volume 4, Issues 1-2, pp. 233 - 262.
- [5] OSTROVSKY E., SIROTA L. *Central Limit Theorem and exponential tail estimation in Hybrid Lebesgue-continuous spaces*. arXiv:1309.2344v1 [math.PR] 9 Sep 2013

- [6] OSTROVSKY E., SIROTA L. *Central Limit Theorem and exponential tail estimation in mixed Lebesgue spaces*. arXiv:1308.5606v1 [math.PR] 26 Aug 2013
- [7] OSTROVSKY E., SIROTA L. *Monte-Carlo method for multiple parametric integrals calculation and solving of linear integral Fredholm equations of a second kind, with confidence regions in uniform norm*. arXiv:1101.5381v1 [math.FA] 27 Jan 2011
- [8] OSTROVSKY E., SIROTA L. *Monte-Carlo computation of multiple weak singular integrals of spherical and Volterra type*. arXiv:1405.6344v1 [math.NA] 24 May 2014
- [9] TALAGRAND M. (1996). Majorizing measure: The generic chaining. *Ann. Probab.* **24** 1049-1103. MR1825156
- [10] TALAGRAND M. (2001). Majorizing Measures without Measures. *Ann. Probab.* **29**, 411-417. MR1825156
- [11] TALAGRAND M. (2005). *The Generic Chaining. Upper and Lower Bounds of Stochastic Processes*. Springer, Berlin. MR2133757.
- [12] TALAGRAND M.(1990). Sample boundedness of stochastic processes under increment conditions.*Ann. Probab.* **18**, 1-49.
- [13] MARCUS M., PISIER G. *Stochastic processes with sample path in exponential Orlich spaces*. Lecture notes in Mathematic, 1153, (1985), 329-358.
- [14] ZHONGGEN SU. *Central limit theorems for random processes with sample paths in exponential Orlicz spaces*. Stochastic Processes and their Applications, **66**, (1997), 1 - 20.
- [15] WEBER M. *Stochastic processes with value in exponential type Orlich spaces*. Ann. of Probab., Vol. 16, No. 3 (Jul., 1988), pp. 1365 - 1371.
- [16] ARAUJO A., GINE E. *The central limit theorem for real and Banach valued random variables*. Wiley, (1980), London, New York.
- [17] BILLINGSLEY P. *Convergence of probability measures*. Wiley, (1968), London, New York.
- [18] DUDLEY R.M. *Uniform Central Limit Theorem*. Cambridge University Press, (1999)
- [19] LEDOUX M., TALAGRAND M. (1991) *Probability in Banach Spaces*. Springer, Berlin, MR 1102015.

- [20] FORTET R. AND MOURIER E. *Les fonctions aléatoires comme éléments aléatoires dans les espaces de Banach*. Studia Math., **15**, (1955), 62-79.
- [21] GARLING D.J.H. *Functional Central Limit Theorems in Banach Spaces*. The Annals of Probability, Vol. 4, No. 4 (Aug., 1976), pp. 600-611
- [22] GINE E. *On the Central Limit theorem for sample continuous processes*. Ann. Probab. (1974), 2, 629-641.
- [23] GINE E., ZINN J. *Central Limit Theorem and Weak Laws of Large Numbers in certain Banach Spaces*. Z. Wahrscheinlichkeitstheorie verw. Gebiete. **62**, (1983), 323-354.
- [24] HEINKEL B. *Measures majorantes et le theoreme de la limite centrale dans $C(S)$* . Z. Wahrscheinlichkeitstheorie. verw. Geb., (1977). 38, 339-351.
- [25] JAIN N.C. AND MARCUS M.B. *Central limit theorem for $C(S)$ valued random variables*. J. of Funct. Anal., (1975), 19, 216-231.
- [26] KOZACHENKO YU. V., OSTROVSKY E.I. (1985). *The Banach Spaces of random Variables of subgaussian type*. Theory of Probab. and Math. Stat. (in Russian). Kiev, KSU, **32**, 43-57.
- [27] OSTROVSKY E., SIROTA L. *CLT for continuous random processes under approximations terms*. arXiv:1304.0250v1 [math.PR] 31 Mar 2013
- [28] OSTROVSKY E., SIROTA L. *Monte Carlo computation of multiple weak singular integrals of spherical and Volterra's type*. arXiv:1405.6344v1 [math.NA] 24 May 2014
- [29] OSTROVSKY E., SIROTA L. *Central Limit Theorem and exponential tail estimations in mixed (anisotropic) Lebesgue spaces*. arXiv:1308.5606v1 [math.PR] 26 Aug 2013
- [30] OSTROVSKY E., SIROTA L. *Central Limit Theorem and exponential tail estimates in hybrid Lebesgue - continuous spaces*. arXiv:1309.2344v1 [math.PR] 9 Sep 2013
- [31] PISIER G., ZINN J. *On the limit theorems for random variables with values in the spaces L_p , $2 \leq p < \infty$* . Z. Wahrscheinlichkeitstheorie verw. Gebiete 41, 289 - 304, (1978).
- [32] RACKAUSKAS A, SUQUET CH. *Central limit theorems in Hölder topologies for Banach space valued random fields*. Teor. Veroyatnost. i Primenen., 2004, Volume 49, Issue 1, Pages 109 - 125 (in Russian).
- [33] ZINN J. *A Note on the Central Limit Theorem in Banach Spaces*. Ann. Probab. Volume 5, Number 2 (1977), 283 - 286.

- [34] FROLOV A.S., TCHENTZOV N.N. *On the calculation by the Monte-Carlo method definite integrals depending on the parameters.* Journal of Computational Mathematics and Mathematical Physics, (1962), V. 2, Issue 4, p. 714-718 (in Russian).
- [35] GRIGORJEVA M.L., OSTROVSKY E.I. *Calculation of Integrals on discontinuous Functions by means of depending trials method.* Journal of Computational Mathematics and Mathematical Physics, (1996), V. 36, Issue 12, p. 28-39 (in Russian).
- [36] OSTROVSKY E., SIROTA L. *Monte-Carlo method for multiple parametric integrals calculation and solving of linear integral Fredholm equations of a second kind, with confidence regions in uniform norm.* arXiv:1101.5381v1 [math.FA] 27 Jan 2011
- [37] OSTROVSKY E. *About supports of probabilistic measures in separable Banach spaces.* Soviet Math., Doklady, (1980), N°6, 836-838.
- [38] CAPONE C., FIORENZA A., KRBEC M. *On the Extrapolation Blowups in the L_p Scale.* Collectanea Mathematica, **48**, 2, (1998), 71-88.
- [39] A.FIORENZA. *Duality and reflexivity in grand Lebesgue spaces.* Collectanea Mathematica (electronic version), **51**, 2, (2000), 131-148.
- [40] A. FIORENZA AND G.E. KARADZHOV. *Grand and small Lebesgue spaces and their analogs.* Consiglio Nazionale Delle Ricerche, Istituto per le Applicazioni del Calcolo Mauro Picone, Sezione di Napoli, Rapporto tecnico n. 272/03, (2005).
- [41] T.IWANIEC AND C. SBORDONE. *On the integrability of the Jacobian under minimal hypotheses.* Arch. Rat.Mech. Anal., 119, (1992), 129 - 143.
- [42] T.IWANIEC, P. KOSKELA AND J. ONNINEN. *Mapping of finite distortion: Monotonicity and Continuity.* Invent. Math. 144 (2001), 507-531.
- [43] OSTROVSKY E., SIROTA L. *Moment Banach spaces: theory and applications.* (2007) HAIT Journal of Science and Engineering C, Volume 4, Issues 1-2, pp. 233 - 262.
- [44] OSTROVSKY E., SIROTA L. *Schlömilch and Bell series for Bessel's functions, with probabilistic applications.* arXiv:0804.0089v1 [math.CV] 1 Apr 2008
- [45] OSTROVSKY E. *Support of Borelian measures in separable Banach spaces.* arXiv:0808.3248v1 [math.FA] 24 Aug 2008

- [46] ROSENTHAL H.P. *On the subspaces of L_p ($p \geq 2$) spanned by sequences of independent Variables.* Israel J. Math., 1970, V.3 pp. 273-253.